CALCULATION OF THE TEMPERATURE FIELD IN A PLATE WITH INTERNAL HEAT SOURCES

G. A. Gemmerling

UDC 536.21

We present a method for calculating the stationary temperature field in a long thin plate with internal heat sources, the plate surface being washed over by a flow of gas. The solution is obtained in the form of a Fourier series with strong convergence.

The calculation of the stationary temperature field in a plate with internal heat sources amounts to integrating the equation

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}=-\frac{\psi}{\lambda}. \tag{1}$$

We assume one side, $y = a$, of the plate to be insulated and the other side, $y = 0$, to be washed over by a nonboiling liquid whose motion we consider to be stationary and the flow homogeneous. If c_p = const, the change in the kinetic energy of the liquid as it moves along the plate is much less than the increase in its heat content, and if there is no heat source in the liquid we can write the energy equation for the liquid in the form

$$\Delta c_p \frac{dT_g}{dx}=q(x). \tag{2}$$

We integrate Eq. (2) under the assumption that

$$\Delta = \text{const}, \quad T_g(0)=T_0. \tag{3}$$

We assume that the heat flow q(x) from the plate to the liquid is convective

$$q(x)=\alpha\,[T(x,\ 0)-T_g(x)], \tag{4}$$

and that the heat transfer coefficient α depends only on the outflow rate Δ.

We write the boundary conditions for Eq. (1) in the form

$$\lambda \left.\frac{\partial T}{\partial y}\right|_{y=0}=q(x), \quad \lambda \left.\frac{\partial T}{\partial y}\right|_{y=a}=0. \tag{5}$$

If $l \gg a$, we can exclude from consideration the boundary conditions at the ends of the plate.

The general statement of the problem is similar to a type of problem given in [1] and the particular solution of it given in [2] (where it was assumed that T_g = const).

We seek a solution of the problem (1)-(5) in the form of a Fourier series. We represent the function q(x), as yet unknown, by its Fourier series:

$$q(x)=a_0+\sum_{n=1}^{\infty}[a_n\cos n\pi x/l+b_n\sin n\pi x/l], \tag{6}$$

We introduce the expansion

$$y^2-2ya=-\frac{2a^2}{3}+\frac{4a^2}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{k^2}\cos k\pi y/a \tag{7}$$

F. E. Dzerzhinskii Military Engineering Academy, Moscow. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 20, No. 3, pp. 411-414, March, 1971. Original article submitted March 10, 1970.

and seek a solution of Eq. (1) in the form

$$T(x,\ y) = -q(x)(y^2 - 2ya)/2a\lambda + A_0 + \sum_{n=1}^{\infty} [A_n \cos n\pi x/l + B_n \sin n\pi x/l]$$
$$+ \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} [A_{nk} \cos n\pi x/l + B_{nk} \sin n\pi x/l] \cos k\pi y/a. \quad (8)$$

We introduce the notation

$$\alpha_n = \left[1 + \frac{1}{3}(an\pi/l)^2\right] \Big/ (n\pi)^2,\ \beta_{nk} = n^2/k^2 \left[\left(\frac{n\pi}{l}\right)^2 + \left(\frac{k\pi}{a}\right)^2\right],$$
$$\gamma_n = \frac{2}{a\lambda}\left(\frac{a}{l}\right)^2 \sum_{k=1}^{\infty} \beta_{nk} - \frac{l^2}{\lambda a}\alpha_n. \quad (9)$$

We now substitute the expansions (6)-(8) into Eqs. (1)-(2) and express the coefficients A_n, B_n, A_{nk}, B_{nk} in terms of a_n, b_n. The final expressions for T(x, y) and T(x, 0) assume the form

$$T(x,\ y) = -\frac{1}{2a\lambda} q(x)(y^2 - 2ya) + A_0 - \sum_{n=1}^{\infty} \frac{\alpha_n l^2}{\lambda a}(a_n \cos n\pi x/l$$
$$+ b_n \sin n\pi x/l) + \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \frac{2}{a\lambda}\left(\frac{a}{l}\right)^2 \beta_{nk}(a_n \cos n\pi x/l + b_n \sin n\pi x/l) \cos k\pi y/a; \quad a_0 = a\psi; \quad (10)$$

$$T(x,\ 0) = A_0 + \sum_{n=1}^{\infty} \gamma_n (a_n \cos n\pi x/l + b_n \sin n\pi x/l). \quad (11)$$

The solution of Eq. (2) with Eq. (6) taken into account is

$$T_g(x) = C_1 + \frac{a_0 x}{D} + \sum_{n=1}^{\infty} \frac{l}{n\pi D}\left[a_n \sin\frac{n\pi x}{l} - b_n \cos\frac{n\pi x}{l}\right],$$
$$C_1 = T_0 + \sum_{n=1}^{\infty} b_n l/n\pi D, \quad D = \Delta c_p. \quad (12)$$

We next express the function a_0x/D in a Fourier series:

$$x = \frac{e_0}{2} + \sum_{n=1}^{\infty} (e_n \cos n\pi x/l + f_n \sin n\pi x/l). \quad (13)$$

In order to obtain a solution with strong convergence [3], we extend the function x onto the interval $[-l, 0]$ so that the function F(x) resulting from this extension will be continuous on the interval $[-l, l]$, along with its first two derivatives. The function F(x) then satisfies the condition

$$F(x) = \begin{cases} x & \text{for} \quad 0 \leqslant x \leqslant l, \\ x - 12x^5/l^4 - 30x^4/l^3 - 20x^3/l^2 & \text{for} \quad -l \leqslant x \leqslant 0. \end{cases} \quad (14)$$

Expanding F(x) in a Fourier series in the interval $(-l, l)$, we obtain

$$e_n = 240l\left[\frac{1}{(n\pi)^4} - \frac{12}{(n\pi)^6}\right], \quad f_n = 0, \quad \text{if} \quad n = 2r+1,\ r = 1,\ 2,\ \ldots,$$
$$e_n = 0, \quad f_n = 1440l\frac{1}{(n\pi)^5}, \quad \text{if} \quad n = 2r. \quad (15)$$

Substituting the expansions (11)-(13) into Eq. (4), we find a_n, b_n:

$$a_n = -g_n f_n, \quad b_n = -h_n f_n \quad \text{for} \quad n = 2r,$$
$$a_n = -h_n e_n, \quad b_n = g_n e_n \quad \text{for} \quad n = 2r+1,$$
$$g_n = \frac{\alpha^2 l a_0 n\pi}{p_n}; \quad h_n = \frac{\alpha a_0 (n\pi D)^2 (1 - \alpha\gamma_n)}{D p_n};$$
$$p_n = n\pi D(1 - \alpha\gamma_n^2) + \alpha^2 l^2. \quad (16)$$

We can now find T(x, y) and $T_g(x)$ from the formulas (10) and (12).

We now examine the convergence of the resulting solution. The coefficients a_n, $b_n \sim n^{-4}$, consequently, the series (6), together with its first two derivatives, converges absolutely and uniformly. Since α_n

~ 1, then A_n, $B_n \sim n^{-4}$; A_{nk} and B_{nk} are of order k^{-4} for fixed n and of order $\sim n^{-4}$ for fixed k. Thus all the series, which arose in the course of obtaining the solution, permit the operations of termwise addition, differentiation, and comparing of coefficients, operations which we have used above.

To calculate the temperature fields from these equations we need to compute the first three to four terms of the series obtained.

The solution given here was obtained for ψ = const. If $\psi = \psi(x)$, then, expanding this function in a Fourier series

$$\psi(x) = s_0 + \sum_{n=1}^{\infty} [s_n \cos n\pi x/l + t_n \sin n\pi x/l] \tag{17}$$

and substituting this expansion into Eq. (1), we obtain corrections to A_n, B_n. The rest of this solution remains unchanged.

Suppose now that we have a heat source ψ_1 in the moving liquid. The corresponding correction is introduced into the right member of Eq. (2). If ψ_1 = const, then only the coefficient of the second term in Eq. (12) is changed. If, however, $\psi_1 = \psi_1(x)$, it is then necessary to represent this function in the form of a series of the type (17). In this case, the second and third terms in Eq. (12) are changed, and also the form of the coefficients (16). However the scheme of the solution stays the same.

NOTATION

x, y	are the coordinates along and across the plate, m;
$T(x, y)$, $T_g(x)$	are the temperatures of the plate and moving liquid (gas), °K;
l	is the plate dimension in liquid flow direction, m;
a	is the thickness of plate, m;
λ	is the thermal conductivity of plate material, W/m · deg;
ψ, ψ_1	are the powers of heat sources in volume of plate and liquid, W/m^3;
c_p	is the heat capacity of liquid, W · sec/kg · deg;
Δ	is the liquid flow rate, kg/m · sec;
α	is the heat transfer coefficient, W/m^2 · deg;
q(x)	is the heat flux through plate–liquid boundary, W/m^2;
T_0	is the initial temperature of liquid, °K.

LITERATURE CITED

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